THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 - Stochastic Processes - 2023/24 Term 2

Solutions to Homework 4

Part 1. Compulsory

1. Solution. Obviously the MC is finite and irreducible, hence it is positive recurrent. Moreover, the chain is aperiodic, hence it has a unique stationary distribution π. Let $\pi = (\pi(0), \pi(1), \pi(2))$, then $\pi P = \pi$ implies that

$$\begin{cases} 0.4\pi(0) + 0.3\pi(1) + 0.2\pi(2) = \pi(0), \\ 0.4\pi(0) + 0.4\pi(1) + 0.4\pi(2) = \pi(1), \\ 0.2\pi(0) + 0.3\pi(1) + 0.4\pi(2) = \pi(2). \end{cases}$$

Together with $\pi(0) + \pi(1) + \pi(2) = 1$, we get $\pi = (\pi(0), \pi(1), \pi(2)) = (0.3, 0.4, 0.3)$.

2. Proof. Suppose that the chain has a stationaty distribution π, then it satisfies $\pi P = \pi$, that is, for any $y \in \mathcal{S}$,

$$\pi(y) = \sum_{x\in\mathcal{S}} \pi(x)P(x,y) = \sum_{x\in\mathcal{S}} \pi(x)\alpha_y = \alpha_y.$$

Also one can check that $\pi(y) = \alpha_y, y \in \mathcal{S}$ satisfies

$$\sum_{y\in\mathcal{S}} \pi(y) = \sum_{y\in\mathcal{S}} \alpha_y = \sum_{y\in\mathcal{S}} P(x,y) = 1.$$

Hence $\pi(y) = \alpha_y, y \in \mathcal{S}$ is the unique stationary distribution.

3. Proof. Note that π satisfies $\pi P^m = \pi$ for any positive integer m. Since x leads to y, there is a positive integer n such that $P^n(x,y) > 0$. Hence

$$\pi(y) = \sum_{z\in\mathcal{S}} \pi(z)P^n(z,y) \geq \pi(x)P^n(x,y) > 0.$$

4. Proof. Note that π satisfies $\pi P = \pi$. Hence

$$\pi(y) = \sum_{x\in\mathcal{S}} \pi(x)P(x,y) = c\sum_{x\in\mathcal{S}} \pi(x)P(x,z) = c\pi(z).$$

5. Proof. **(a)** Clearly $\pi_\alpha(x) \geq 0$ for $x \in \mathcal{S}$ and

$$\sum_{x\in\mathcal{S}} \pi_\alpha(x) = (1-\alpha)\sum_{x\in\mathcal{S}} \pi_0(x) + \alpha\sum_{x\in\mathcal{S}} \pi_1(x) = (1-\alpha) + \alpha = 1.$$

Moreover, we have for any $y \in \mathcal{S}$,

$$\begin{aligned}
(\pi_\alpha P)(y) &= \sum_{x \in \mathcal{S}} \pi_\alpha(x) P(x,y) \\
&= \sum_{x \in \mathcal{S}} ((1-\alpha)\pi_0(x) + \alpha\pi_1(x)) P(x,y) \\
&= (1-\alpha) \sum_{x \in \mathcal{S}} \pi_0(x) P(x,y) + \alpha \sum_{x \in \mathcal{S}} \pi_1(x) P(x,y) \\
&= (1-\alpha)\pi_0(y) + \alpha\pi_1(y) = \pi_\alpha(y).
\end{aligned}$$

Hence π_α is a stationary distribution.

(b) Since π_0 and π_1 are distinct, we can choose $x_0 \in \mathcal{S}$ such that $\pi_0(x_0) \neq \pi_1(x_0)$. If $\alpha \neq \beta \in [0,1]$, then

$$\pi_\alpha(x_0) - \pi_\beta(x_0) = (\alpha - \beta)(\pi_1(x_0) - \pi_0(x_0)) \neq 0.$$

Hence $\pi_\alpha \neq \pi_\beta$.

6. Solution. The transition matrix is given by

$$P = \begin{pmatrix} 0\,1\,0\,0\,0 \cdots \\ q\,0\,p\,0\,0 \cdots \\ 0\,q\,0\,p\,0 \cdots \\ 0\,0\,q\,0\,p \cdots \\ \vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\ddots \end{pmatrix}.$$

Suppose that the stationary distribution π exists. Then by $\pi P = \pi$,

$$\begin{aligned}
\pi(1)q = \pi(0) \quad &\Rightarrow \quad \pi(1) = \frac{1}{q}\pi(0), \\
\pi(0) + \pi(2)q = \pi(1) \quad &\Rightarrow \quad \pi(2) = \frac{\pi(1) - \pi(0)}{q} = \frac{p}{q^2}\pi(0), \\
\pi(1)p + \pi(3)q = \pi(2) \quad &\Rightarrow \quad \pi(2) = \frac{\pi(2) - p\pi(1)}{q} = \frac{p^2}{q^3}\pi(0), \\
&\cdots
\end{aligned}$$

By induction, $\pi(n) = \dfrac{\pi(0)}{p}\left(\dfrac{p}{q}\right)^n$, $n \geq 1$.

If $p \geq q$ (i.e. $p \geq 1/2$), then $\sum_{n=1}^{\infty} \pi(n) \geq \frac{1}{p}\sum_{n=1}^{\infty} \pi(0) = \infty$. Thus, the stationary distribution does not exist.

On the other hand, if $p < q$ (i.e. $p < 1/2$), we have

$$\sum_{n=0}^{\infty} \pi(n) = \left(1 + \frac{1}{p}\sum_{n=1}^{\infty}\left(\frac{p}{q}\right)^n\right)\pi(0) = \frac{2(1-p)}{1-2p}\pi(0).$$

Hence the unique stationary disrtibution is given by

$$\pi(0) = \frac{1-2p}{2(1-p)}, \quad \pi(n) = \frac{1-2p}{2(1-p)p}\left(\frac{p}{1-p}\right)^n, n \geq 1.$$

10. Solution. Since X_0 has the stationary distribution π, X_1 also has the stationary distribution π. Note that

$$P(X_0 = y \mid X_1 = x) = \frac{P(X_0 = y, X_1 = x)}{P(X_1 = x)} = \frac{\pi(y)P(y,x)}{\pi(x)}.$$

It suffices to show that for any $x, y \in \mathcal{S}$, $\pi(x)P(x,y) = \pi(y)P(y,x)$. For $y = x$ or $|y-x| \geq 2$, the equation is trivial. If $y = x+1$, then by (9),

$$\pi(x)P(x,x+1) = \pi(0)\pi_x p_x = \pi(0)\frac{p_0 \cdots p_{x-1}p_x}{q_1 \cdots q_x} = \pi(0)\pi_{x+1}q_{x+1} = \pi(x+1)P(x+1,x).$$

If $y = x-1$, $x \geq 1$, then by (9),

$$\pi(x)P(x,x-1) = \pi(0)\pi_x q_x = \pi(0)\frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_{x-1}} = \pi(0)\pi_{x-1}p_{x-1} = \pi(x-1)P(x-1,x).$$

Part 2. **Optional**

7. Solution. (a) The transition matrix is given by

$$P = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdots & d-2 & d-1 & d \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{d} & 0 & \frac{d-1}{d} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{2}{d} & 0 & \frac{d-2}{d} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{d} & 0 & \frac{d-3}{d} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{d-1}{d} & 0 & \frac{1}{d} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Let π be the stationary distribution. Then by $\pi P = \pi$,

$$\begin{aligned} \pi(1)\frac{1}{d} &= \pi(0) \Rightarrow \pi(1) = d\pi(0) = \tbinom{d}{1}\pi(0), \\ \pi(0) + \pi(2)\frac{2}{d} &= \pi(1) \Rightarrow \pi(2) = \frac{d(d-1)\pi(0)}{2} = \tbinom{d}{2}\pi(0), \\ \pi(1)\frac{d-1}{d} + \pi(3)\frac{3}{d} &= \pi(2) \Rightarrow \pi(3) = \frac{d(d-1)(d-2)\pi(0)}{6} = \tbinom{d}{3}\pi(0), \\ &\cdots \end{aligned}$$

By induction, $\pi(n) = \binom{d}{n}\pi(0)$, $0 \leq n \leq d$. Together with $\sum_{n=0}^{d} \pi(n) = 1$, the stationary distribution must be

$$\pi(n) = \frac{\binom{d}{n}}{2^d}, \quad 0 \leq n \leq d.$$

(b) The mean of this distribution is given by

$$\sum_{x=0}^{d} x\frac{\binom{d}{x}}{2^d} = \frac{1}{2^d}\sum_{x=0}^{d} x\binom{d}{x} = \frac{d}{2^d}\sum_{x=1}^{d}\binom{d-1}{x-1} = \frac{d}{2^d}2^{d-1} = \frac{d}{2}.$$

Note that

$$\begin{aligned}\sum_{x=0}^{d} x^2\binom{d}{x} &= \sum_{x=2}^{d} x(x-1)\binom{d}{x} + \sum_{x=1}^{d} x\binom{d}{x}\\ &= d(d-1)\sum_{x=2}^{d}\binom{d-2}{x-2} + d\sum_{x=1}^{d}\binom{d-1}{x-1}\\ &= d(d-1)2^{d-2} + d2^{d-1}.\end{aligned}$$

Hence, the variance is given by

$$\sum_{x=0}^{d} x^2\frac{\binom{d}{x}}{2^d} - \left(\sum_{x=0}^{d} x\frac{\binom{d}{x}}{2^d}\right)^2 = \frac{d(d-1)}{4} + \frac{d}{2} - \left(\frac{d}{2}\right)^2 = \frac{d}{4}.$$

8. Proof. The transition matrix is given by

$$P = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdots & d-2 & d-1 & d \end{matrix} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2d} & \frac{1}{2} & \frac{d-1}{2d} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{2}{2d} & \frac{1}{2} & \frac{d-2}{2d} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2d} & \frac{1}{2} & \frac{d-3}{2d} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{d-1}{2d} & \frac{1}{2} & \frac{1}{2d} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

Let π be the stationary distribution. Then by $\pi P = \pi$,

$$\begin{aligned} \pi(1)\frac{1}{2d} &= \frac{\pi(0)}{2} &&\Rightarrow \pi(1) = d\pi(0) = \tbinom{d}{1}\pi(0),\\ \frac{\pi(0)}{2} + \pi(2)\frac{2}{2d} &= \frac{\pi(1)}{2} &&\Rightarrow \pi(2) = \frac{d(d-1)\pi(0)}{2} = \tbinom{d}{2}\pi(0),\\ \pi(1)\frac{d-1}{2d} + \pi(3)\frac{3}{2d} &= \frac{\pi(2)}{2} &&\Rightarrow \pi(3) = \frac{d(d-1)(d-2)\pi(0)}{6} = \tbinom{d}{3}\pi(0),\\ &\cdots \end{aligned}$$

By induction, $\pi(n) = \binom{d}{n}\pi(0)$, $0 \le n \le d$. Together with $\sum_{n=0}^{d}\pi(n) = 1$, the stationary distribution must be

$$\pi(n) = \frac{\binom{d}{n}}{2^d}, \quad 0 \le n \le d.$$

The result is the same as the one of the original Ehrenfest chain.

9. Solution. Let π be the stationary distribution. The transition function is given by

$$P(x,y) = \begin{cases} q_x = \left(\frac{x}{d}\right)^2, & \text{if } y = x-1, x \neq 0;\\ r_x = 2\left(\frac{x}{d}\right)\left(\frac{d-x}{d}\right), & \text{if } y = x;\\ p_x = \left(\frac{d-x}{d}\right)^2, & \text{if } y = x+1, x \neq d;\\ 0, & \text{otherwise.} \end{cases}$$

We can apply the result in page 51 of the textbook, for $x \geq 1$,

$$\pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{d^2(d-1)^2 \cdots (d-x+1)^2}{(x!)^2} = \binom{d}{x}^2,$$

and set $\pi_0 = 1 = \binom{d}{0}$. By the hint,

$$\pi(0) = \frac{1}{\sum_{x=0}^{d} \pi_x} = \frac{1}{\binom{2d}{d}} = \frac{\binom{d}{0}^2}{\binom{2d}{d}}.$$

Hence $\pi(x) = \pi_x \pi(0) = \frac{\binom{d}{x}^2}{\binom{2d}{d}}$, $0 \leq x \leq d$.